UDC 531.36: 534

UNSAFE STABILITY BOUNDARIES OF THE LORENTZ MODEL

PMM Vol. 42, No. 5, 1978, pp 950-952
N. V. ROSHCHIN
(Gor'kii)
(Received January 30, 1978)

Properties of the Lorentz model stability boundaries are investigated by methods of the qualitative theory of differential equations developed by Bautin [1]. It is shown that for positive physical parameters the boundary of the stability region is unsafe.

The Lorentz model [2] is defined by the system of equations

$$x^{\cdot} = -\sigma(x - y),\ y^{\cdot} = -xz + rx - y,\ z^{\cdot} = xy - bz \tag{1}$$

where b, σ, and r are positive physical parameters, to which attention was drawn in connection with the problem of existence of stable limit sets of complex nature, viz. the so-called "strange attractors" [3, 4], and the ensuing new interpretation of turbulence. System (1) can have three stability states

$$O_1\ (x = y = z = 0),\quad O_2\ (x = y = +[b(r-1)]^{1/2},\ z = r - 1)$$
$$O_3\ (x = y = -[b(r-1)]^{1/2},\ z = r - 1)$$

The equilibrium state O_1 exists for any values of parameters and is a stable node when $r < 1$ or a saddle when $r > 1$. When $r > 1$ the bifurcation of the equilibrium state O_1 $(r = 1)$ results in the appearance of stable equilibrium state O_2 and O_3. Properties of the boundaries of equilibrium stability region of states O_2 and O_3 are investigated below in the critical case of two pure imaginary roots of the characteristic equation. Interest in this problem was stimulated by new results [4] which indicate the existence of a strict mode of the onset of stochastic properties and hysteresis for $\sigma = 10$, $b = {}^8/_3$ and $r = 24.74$, which results in the loss of stability of equilibrium states O_2 and O_3.

The Routh — Hurwitz conditions imply that the stability boundary in the parameter space of the equilibrium states O_2 and O_3 is determined by the equation

$$S = b(\sigma + b + 1)(r + \sigma) - 2\sigma b(r - 1) = 0 \tag{2}$$

When the stability boundary is passed from positive to negative values of S, the nature of equilibrium states O_2 and O_3 changes from a stable focus to a saddle-focus, while at the boundary itself the characteristic equation for these equilibrium states has two pure imaginary roots. In that case all variants of phase trajectory behavior in the neighborhood of the equilibrium state had been thoroughly investigated; they are determined by the sign of the Liapunov or focal quantities [1, 5].

The analysis carried out below makes it possible to assert that for positive parameters the Liapunov quantity is positive, i.e. the stability boundary of the Lorentz model is unsafe. It should be noted that the estimate of the sign of Liapunov's quantity obtained in [6] for the limit case $\sigma \to \infty$ conforms to that assertion. (Numerical calculations by York and Ruelle cited in [7] according to which the Liapunov quantity may change its sign when the parameters are positive, is not corroborated by the direct determination on a computer of phase trajectories near the equilibrium stability boundary; detailed calculations were carried out for the following values of parameters: $\sigma = 100$, $b = 75$, $r \approx 741.66$ and for $\sigma = 100$, $b = 1$, $r \approx 106.12$. The above statement is also valid in the case of the mathematically similar problem of the single-mode quantum generator [4, 8, 9].

An exhaustive and fairly convenient for practical applications algorithm for the determination the first Liapunov quantity in terms of coefficients of the input system (1) was formulated by Bautin [1]. That algorithm is used below.

To obtain the simplest expression for the Liapunov quantity we represent system (1) in the form

$$\begin{aligned} x_1^{\cdot} &= -(\sigma + b + 1)\, x_1 - b\,(1 + \sigma)\, x_2 + b\sigma\,(r - 1)\, x_3 + x_1 x_2 / x_3 + \\ &\quad (\sigma + 1)\, x_2^2 / x_3 - x_3^2 x_2 - \sigma x_3^3 = P_1(x_1, x_2, x_3) \\ x_2^{\cdot} &= x_1 = P_2(x_1, x_2, x_3), \quad x_3^{\cdot} = x_2 = P_3(x_1, x_2, x_3) \\ x_1 &= x^{\cdot\cdot}, \quad x_2 = x^{\cdot}, \quad x_3 = x \end{aligned} \tag{3}$$

Then in conformity with [1] we reduce system (3) to the standard form by expanding $P_i(x_1, x_2, x_3)$, $i = 1, 2, 3$ in series in powers of the new variables $x_j' = x_j - x_j^{\circ}$, $j = 1, 2, 3$. The term x_j° denotes coordinates of one of the equilibrium states O_2 and O_3. Limiting the expansion to and including third order terms and omitting primes at new variables, we obtain the following system:

$$\begin{aligned} x_j^{\cdot} &= a_1^{(j)} x_1 + a_2^{(j)} x_2 + a_3^{(j)} x_3 + a_{11}^{(j)} x_1^2 + a_{22}^{(j)} x_2^2 + a_{33}^{(j)} x_3^2 + \\ &\quad 2a_{12}^{(j)} x_1 x_2 + 2a_{13}^{(j)} x_1 x_3 + 2a_{23}^{(j)} x_2 x_3 + a_{111}^{(j)} x_1^3 + a_{222}^{(j)} x_2^3 + a_{333}^{(j)} x_3^3 + \\ &\quad 3a_{112}^{(j)} x_1^2 x_2 + 3a_{113}^{(j)} x_1^2 x_3 + 3a_{122}^{(j)} x_1 x_2^2 + 3a_{223}^{(j)} x_2^2 x_3 + 3a_{133}^{(j)} x_1 x_3^2 + \\ &\quad 3a_{233}^{(j)} x_2 x_3^2 + 6a_{123}^{(j)} x_1 x_2 x_3, \quad j = 1, 2, 3 \end{aligned} \tag{4}$$

whose nonzero coefficients are

$$\begin{aligned} & a_1^{(1)} = -p = -\sigma - b - 1, \quad a_2^{(1)} = -q = -b\,(\sigma + r), \quad a_3^{(1)} = \\ &\quad -2b\sigma\,(r - 1) \\ & a_{22}^{(1)} = (\sigma + 1)\,[b\,(r - 1)]^{-1/2}, \quad a_{33}^{(1)} = -3\sigma[b(r - 1)]^{1/2}, \\ & a_{12}^{(1)} = {}^{1}/_{2}\,[b\,(r - 1)]^{-1/2} \\ & a_{23}^{(1)} = -[b\,(r - 1)]^{1/2}, \quad a_{333}^{(1)} = -\sigma, \quad a_{223}^{(1)} = -(\sigma + 1)\,[3b\,(r - 1)]^{-1} \\ & a_{233}^{(1)} = -{}^{1}/_{3}, \; a_{123}^{(1)} = -{}^{1}/_{6}\,[b(r - 1)]^{-1}, \; a_1^{(2)} = 1, \; a_2^{(3)} = 1 \end{aligned} \tag{5}$$

For the first Liapunov quantity we have the following expression:

$$\begin{aligned} g &= {}^{1}/_{4}\pi\,[2\,(A_{33}^{(3)} A_{33}^{(3)} - A_{22}^{(2)} A_{22}^{(3)}) + 2A_{23}^{(2)}\,(A_{22}^{(2)} + A_{33}^{(2)}) - 2A_{23}^{(3)}(A_{22}^{(3)} + A_{33}^{(3)}) + \\ &\quad 3q^{1/2}\,(A_{222}^{(2)} + A_{333}^{(3)} + A_{233}^{(2)} + A_{223}^{(3)})] + {}^{1}/_{4}\pi\,[pq^{1/2}\,(p^2 + 4q)]^{-1} \times \\ &\quad \{p^3\,[2A_{22}^{(1)}\,(3A_{12}^{(2)} + A_{13}^{(3)}) + 2A_{33}^{(1)}\,(A_{12}^{(2)} + 3A_{13}^{(3)})] + 4A_{23}^{(1)}\,(A_{13}^{(2)} + A_{12}^{(3)})] + \\ &\quad 4pq^{1/2}\,[(A_{22}^{(1)} - A_{33}^{(1)})\,(A_{13}^{(2)} + A_{12}^{(3)}) + 2A_{23}^{(1)}\,(A_{13}^{(3)} - A_{12}^{(2)})] + \\ &\quad 16q\,(A_{22}^{(1)} + A_{33}^{(1)})\,(A_{12}^{(2)} + A_{13}^{(3)})\} \end{aligned} \tag{6}$$

At the stability boundary $A_{kl}^{(j)}$ and $A_{kls}^{(j)}$ are defined in terms of coefficients of system (4) by formulas

$$A_{12}^{(j)} = \frac{\alpha_{j1} q}{\Delta_0} (a_{33}^{(1)} - p a_{23}^{(1)} + pq a_{12}^{(1)}), \quad A_{13}^{(j)} = \frac{\alpha_{j1}' q^{3/2}}{\Delta_0} (p a_{22}^{(1)} - a_{23}^{(1)} - p^2 a_{12}^{(1)}),$$

$$A_{23}^{(j)} = \frac{\alpha_{j1}' q^{1/2}}{\Delta_0} (q a_{12}^{(1)} - a_{23}^{(1)})$$

$$A_{22}^{(j)} = \frac{\alpha_{j1}'}{\Delta_0} a_{33}^{(1)}, \quad A_{33}^{(j)} = \frac{\alpha_{j1}' q}{\Delta_0} a_{22}^{(1)},$$

$$A_{222}^{(2)} = \frac{\alpha_{21}'}{\Delta_0} a_{333}^{(1)}, \quad A_{233}^{(2)} = \frac{\alpha_{21}' q}{\Delta_0} a_{223}^{(1)},$$

$$A_{333}^{(3)} = 0, \quad A_{223}^{(3)} = \frac{\alpha_{31}' q^{1/2}}{\Delta_0} (-a_{223}^{(1)} + 2q a_{123}^{(1)})$$

$$\alpha_{11}' = q^{1/2}, \quad \alpha_{21}' = -q^{3/2}, \quad d_{31}' = -pq, \quad \Delta_0 = q^{3/2} (p^2 + q), \quad j = 1, 2, 3$$

Substituting expressions (5) and (6) and taking into account formula (2), for the first Liapunov quantity at the stability limit S we obtain the final formula

$$g = {}^1/_2 b\pi [p q^{1/2} (p^2 + q)(p^2 + 4q)(\sigma - b - 1)]^{-1} [9\sigma^4 + (-18b + 20)\sigma^3 + (20b^2 + 2b + 10)\sigma^2 + (-2b^3 + 12b^2 + 10b - 4)\sigma - b^4 - 6b^3 - 12b^2 - 10b - 3] \tag{7}$$

This formula considerably simplifies the determination of the sign of Liapunov's quantity for any specified parameter values at the stability boundary. It does not, however, allow the establishment of the necessary general statements about this problem. To overcome this shortcoming we use some simple reasoning which leads to a simpler form of formula (7).

From the physical aspect of this problem of interest is that section of the stability boundary in which all three parameters b, σ, and r are positive. Since at the stability boundary the parameter $r = \sigma(\sigma + b + 3)(\sigma - b - 1)^{-1}$, it is sufficient to consider the region of parameters σ and b defined by the condition $0 < b < \sigma - 1$. The substitution of $\sigma = \sigma_* + b + 1$ into (7) then shows that the first Liapunov quantity g is positive when $b > 0$ and $\sigma_* > 0$. This confirms the assertion about the unsafe character of the stability region boundary of the Lorentz model.

The author thanks L. P. Shil'nikov for his interest in this work.

REFERENCES

1. Bautin, N. N., Behavior of Dynamic Systems in Proximity of Stability Region Boundaries. Leningrad—Moscow, Gostekhizdat, 1949.

2. Lorentz, E. N., Deterministic nonperiodic flow. J. Atmosph. Sci., Vol. 20, No. 2, 1963.

3. Ruelle, D., Strange attractors as a mathematical explanation of turbulence. Lect. Notes Phys., Vol. 12, 1975.

4. Afraimovich, V. S., Bykov, V. V., and Shilnikov, L. P., On the emergence in the structure of the Lorentz attractor. Dokl. Akad. Nauk, SSSR, Vol. 234, No. 2, 1977.

5. Shil'nikov, L. P., Theory of bifurcation of dynamic systems and dangerous limits. Dokl. Akad. Nauk, SSSR, Vol. 224, No. 5, 1975.

6. McLaughlin, J. B. and Martin, P. C., Transition to turbulence in a statically stressed fluid system. Phys. Rev., Ser. A, Vol. 12, No. 1, 1975.

7. Marsden, J. E. and McCracken, M., The Hopf Bifurcation and its Applications. New York, Springer Verlag, 1976.

8. Oravskii, A. N. and Uspenskii, A. V., The pulsation mode of radiation power of quantum generators. Tr. Fiz. Inst. Akad. Nauk. SSSR. Kvantovaia Radiofizika, Vol. 31, 1965.

9. Haken, H., Analogy between higher instabilities in fluids and lasers. Phys. Letters, Ser.A., Vol. 53, No. 1, 1975.

Translated by J.J. D.